MTH 2165. Week 3 Friday. Lecture Notes.

Power Series and Power Series Representations.

Definition 1. The Power Series.

A series of the form $\sum_{n=0}^{\infty} c_n x^n$ for some sequence (c_n) is called a power series.

Here, the series is evaluated point-wise. That is, for each $x \in \mathbb{R}$, $\sum_{n=0}^{\infty} c_n x^n$ is a separate series.

A series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is called a power series centered at $x=a$;

The Question is: For which values of $x \in \mathbb{R}$ does $\sum_{n=0}^{\infty} c_n x^n$ converge?

Example 1.1 Let $\sum_{n=0}^{\infty} x^n$; This is a geometric series with $r = x$. Then, $\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$.

Example 1.2. Consider $\sum_{n=0}^{\infty} n!\,x^n$; $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = \lim_{n\to\infty} |(n+1)x|$;

If $x = 0$, then $L = 0$ and $\sum n!\,x^n$ converges by the Ratio Test.

If $x \neq 0$, then $L = \lim_{n\to\infty} (n+1)|x| = \infty$ and $\sum n!\,x^n$ diverges by the Ratio Test.

Algebra Review: Let $f(x)$ be any function and $k \in \mathbb{R}$ be positive;

Then, ① $|f(x)| = k \iff f(x) = k$ or $f(x) = -k$;

② $|f(x)| < k \iff -k < f(x) < k$;

③ $|f(x)| > k \iff f(x) < -k$ or $f(x) > k$;

Example 1.3. For which $x \in \mathbb{R}$ does $s(x) = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Using the Ratio Test: $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = \lim_{n\to\infty} |x-3| = |x-3|$;

The value of L depends on x:

(1) If $|x-3| < 1$: Then, $L < 1$ and $s(x)$ absolutely converges by the Ratio Test. Here, $x \in (2,4)$;

(2) If $|x-3| > 1$: Then, $L > 1$ and $s(x)$ diverges by the Ratio Test. Here, $x \in (-\infty, 2) \cup (4, \infty)$;

(3) If $|x-3| = 1$, then $L = 1$ and the Ratio Test is inconclusive; Use another test.

Case (i): $x - 3 = 1$ and $x = 4$;

$$s(4) = \sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges as a p-series with } p = 1 \leq 1;$$

Case (ii): $x - 3 = -1$ and $x = 2$;

$$s(2) = \sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges since it's the alternating harmonic series.}$$

$\therefore \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converges if and only if $x \in [2,4)$;

Example 1.4. The Bessel Function of Order 0: $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$; Find all $x \in \mathbb{R}$ s.t. $J_0(x)$ converges.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{2^{2n+2}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2 (n!)^2}{2(n+1)^2 (n!)^2} \right| = \lim_{n\to\infty} \left| \frac{x^2}{2(n+1)^2} \right| = 0 ;$$

Since $L < 1$ for all $x \in \mathbb{R}$, $J_0(x)$ converges for all $x \in \mathbb{R}$ by the Ratio Test.

Proposition 2. Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at $x=a$.

Exactly one of the following is true about the convergence of $\sum c_n (x-a)^n$:

1. $\sum c_n (x-a)^n$ converges only for $x=a$;
2. $\sum c_n (x-a)^n$ converges for all $x \in \mathbb{R}$;
3. There exists $R \in \mathbb{R}$ such that $\sum c_n (x-a)^n$ converges if $|x-a| < R$ and $\sum c_n (x-a)^n$ diverges if $|x-a| > R$;

   Note that $\sum c_n (x-a)^n$ may converge or diverge if $|x-a| = R$, i.e. $x = R+a$ or $x = -R+a$;

Definition 3. Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series. We define the radius R and interval I of convergence as:

1. If $\sum c_n (x-a)^n$ converges only for $x=a$: $R=0$ and $I = \{a\}$;
2. If $\sum c_n (x-a)^n$ converges for all $x \in \mathbb{R}$: $R = \infty$ and $I = (-\infty, \infty) = \mathbb{R}$.
3. If there exists $R \in \mathbb{R}$ such that $\sum c_n (x-a)^n$ converges if $|x-a| < R$ and $\sum c_n (x-a)^n$ diverges if $|x-a| > R$,

   then R is called the radius of convergence and

   I is one of $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R]$;

Example 3.1. The series $\sum_{n=0}^{\infty} x^n$ has a radius of convergence $R=1$ and interval of convergence $(-1,1)$;

Example 3.2. The series $\sum_{n=0}^{\infty} n!\, x^n$ has $R=0$ and $I = \{0\}$;

Example 3.3. The series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ has $R=1$ and $I = [2,4)$;

Example 3.4. The series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ has $R = \infty$ and $I = (-\infty, \infty) = \mathbb{R}$;

Definition 4. A function $f(x)$ admits a power series representation (p.s.r.) $\sum_{n=0}^{\infty} c_n (x-a)^n$ if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on some interval $I \subseteq \mathbb{R}$;

Remark: This is not yet the Taylor series.

Example 4.1. Recall that $\sum_{n=0}^{\infty} ar^n = \frac{ar^0}{1-r} = \frac{a}{1-r}$; Applying this to $\sum_{n=0}^{\infty} x^n$, we get:

the geometric power series $g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$;

Proposition 5. Let $f(x)$ admit a power series representation $f(x)=\sum_{n=0}^{\infty} c_n x^n$. Let $p(x)$ be a polynomial.

① Function composition: $f(p(x))$ admits a p.s.r. $f(p(x))=\sum_{n=0}^{\infty} c_n (p(x))^n$;

② Multiplication: $p(x)f(x)$ admits a p.s.r. $p(x)f(x)=\sum_{n=0}^{\infty} c_n\, p(x)\, x^n$;

The p.s.r. of $f(p(x))$ and $p(x)f(x)$ may not have the same interval of converges as that of $f(x)$;

For the examples below, let $g(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$ with interval of convergence $(-1,1)$;

Example 5.1. Find a p.s.r. for $f(x)=\frac{1}{1+x}$; Since $f(x)=g(-x)$, then $f(x)=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n$;

Since $g(x)$ has interval of convergence $\{x: |x|<1\}$,
then $f(x)$ has interval of convergence $\{x: |-x|=|x|<1\}=(-1,1)$;

Example 5.2. Find a p.s.r. for $h(x)=\frac{1}{1+x^2}$; Since $h(x)=g(-x^2)$, then $h(x)=\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}$
with interval of convergence $|-x^2|=|x|^2<1$; $|x|<1$;

Example 5.3. Find a p.s.r. for $a(x)=\frac{x}{1-x}$; Since $a(x)=xf(x)$, then $a(x)=\sum_{n=0}^{\infty}(x)(x^n)=\sum_{n=0}^{\infty}x^{n+1}$;
with interval of convergence $|x|<1$ or $(-1,1)$;

Example 5.3. Find a p.s.r. for $b(x)=\frac{1}{x+2}$;

$$b(x)=\frac{1}{x+2}=\frac{1}{2(1+\frac{x}{2})}=\frac{1}{2}\left(\frac{1}{1-(-\frac{x}{2})}\right)=\frac{1}{2}f(-\tfrac{x}{2})=\sum_{n=0}^{\infty}\frac{1}{2}(-\tfrac{x}{2})^n=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}};$$

The series converges if $|-\frac{x}{2}|=\frac{1}{2}|x|<1$; $|x|<2$; The interval of convergence is $(-2,2)$;

Theorem 6. Term-by-Term Differentiation + Integration.

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series with radius of convergence $R>0$; Let $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$;

Then, ① $f(x)$ is differentiable on $(a-R, a+R)$;

② $$f'(x)=\frac{d}{dx}\left[\sum_{n=0}^{\infty}c_n(x-a)^n\right]=\sum_{n=0}^{\infty}\frac{d}{dx}\left[c_n(x-a)^n\right]=\sum_{n=1}^{\infty}c_n n(x-a)^{n-1};$$

③ $$\int f(x)\,dx=\int\left[\sum_{n=0}^{\infty}c_n(x-a)^n\right]dx=\sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx=C+\sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1};$$

with the power series of $f'(x)$ and $\int f(x)\,dx$ having same radius of convergence as that of $f(x)$ but not necessarily the same interval of convergence.

For the examples below, let $g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with radius of convergence $R=1$ and interval of convergence $(-1,1)$;

Example 6.1. Find a p.s.r. for $f(x) = \frac{1}{(1-x)^2}$;

Since $g'(x) = \frac{1}{(1-x)^2} = f(x)$, $f(x) = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=1}^{\infty} n x^{n-1}$ with $R = 1$;

Example 6.2. Find a p.s.r. for $h(x) = \ln(1+x)$;

Since $h'(x) = \frac{1}{1+x} = \frac{1}{1+(-x)} = g(-x)$ and $h(x) = \int g(-x)\,dx$,

$$h(x) = \sum_{n=0}^{\infty} \int (-x)^n \, dx = \sum_{n=0}^{\infty} \int (-1)^n x^n \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1};$$

To find C, plug-in $x=0$: $h(0) = \ln(1+0) = 0 = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(0)^{n+1} = C$; $\therefore C = 0$;

$\therefore h(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ with $R = 1$;

Example 6.3. (a) Evaluate $\int \frac{1}{1+x^7}\,dx$ as a power series; (b) Approximate $\int_0^{0.5} \frac{1}{1+x^7}\,dx$ to within 10^{-7};

Part (a). Let $F(x) = \int \frac{1}{1+x^7}\,dx$; Then, $F(x) = \int g(-x^7)\,dx = \int \sum_{n=0}^{\infty} (-x^7)^n \, dx$

$$= \sum_{n=0}^{\infty} \int (-1)^n x^{7n} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1} x^{7n+1}$$ for some $C \in \mathbb{R}$ with $R = 1$

since $\sum_{n=0}^{\infty} (-1) x^{7n}$ has $R = 1$;

Part (b). $A = \int_0^{0.5} \frac{1}{1+x^7}\,dx = F(0.5) - \cancel{F(0)}^{0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1}(0.5)^{7n+1}$; This is an alternating series.

Let $b_n = \frac{1}{7n+1}\left(\frac{1}{2}\right)^{7n+1}$; Then, b_n is positive and decreasing for $n \geq 1$ and $\lim_{n\to\infty} \frac{1}{7n+1}\left(\frac{1}{2}\right)^{7n} = 0$;

$\therefore$ Alternating Series Approximation Theorem applies and $|A - S_N| < b_{N+1}$

where S_N is the N^{th} partial sum of $\sum b_n$;

We can find N s.t. $b_{N+1} < 10^{-7}$ by brute force: $N=2$: $b_3 = 2.03 \times 10^{-6}$;

$N=3$: $b_4 = 1.08 \times 10^{-8}$; ← This is good.

$\therefore \sum_{n=0}^{3} \frac{(-1)^n}{7n+1}\left(\frac{1}{2}\right)^{7n+1} = 0.49951374 \approx A$ within an error of 10^{-7};